MR1658220 (99i:14065) 14N10 (14H40 14J28)
Fantechi, B. (S-MLI); **Göttsche, L.** (I-ICTP); **van Straten, D.** (D-MNZ)
Euler number of the compactified Jacobian and multiplicity of rational curves. (English summary)
J. Algebraic Geom. **8** (1999), *no. 1,* 115–133.

Rational curves on $K3$ surfaces are finite in number and the number of rational curves of a given degree is given by the coefficients of the generating function

$$q/\Delta(q)=\prod_{m=1}^{\infty}(1-q^m)^{-24}$$

where Δ is a well-known modular form of weight 12.

This remarkable formula was discovered by S.-T. Yau and E. Zaslow [Nuclear Phys. B **471** (1996), no. 3, 503–512; MR1398633 (97e:14066)] using ideas from physics. In their paper, they outlined a mathematical strategy for counting rational curves and deriving the above formula. This strategy was carried out by A. Beauville [Duke Math. J. **97** (1999), no. 1, 99–108 MR1682284 (2000c:14073)].

Intrinsic in the method of Yau-Zaslow (carried out by Beauville) is that one must count rational curves with multiplicities. The multiplicity of a rational curve C is defined to be the Euler characteristic of the compactified Jacobian $\overline{J}C$ whose points parameterize rank-one torsion-free sheaves of degree zero on C. Note that this multiplicity is 1 if the singularities of C are nodal.

The paper under review relates the multiplicities of rational curves defined above to multiplicities that arise naturally in Gromov-Witten theory and stable maps. The normalization map $\tilde{C}\to C$ can be regarded as a stable map to the $K3$ surface X by composition with the inclusion $C\subset X$, and it defines a 0-dimensional subscheme of the moduli space of genus-zero stable maps to X. In this paper the authors prove that the length of this subscheme coincides with the multiplicity defined by the compactified Jacobian of C.

The length of the normalization map in the moduli space of stable maps occurs as the contribution of C to a modified Gromov-Witten invariant. Modifying the usual Gromov-Witten invariants and using them to count curves on $K3$ surfaces was done by the reviewer and N. C. Leung ["The enumeration geometry of $K3$ surfaces and modular forms", Preprint, http://xxx.lanl.gov/abs/alg-geom/9711031]. The modified Gromov-Witten invariant methods apply to counting curves of arbitrary genus in a $K3$, and we prove that the corresponding generating function for genus g is $(DG_2)^g/\Delta$, as was conjectured by L. Göttsche [Comm. Math. Phys. **196** (1998), no. 3, 523–533 MR1645204 (2000f:14085)].

The result of the paper under review is actually more general than the application to rational curves in $K3$ surfaces discussed above. The authors prove that for any rational curve with locally planar singularities, the Euler characteristic of the compactified Jacobian is equal to the multiplicity of the $\delta=$ constant stratum in the base of a semi-universal deformation of C.

Reviewed by *Jim A. Bryan*

References

1. A. Altman, A. Iarrobino, and S. Kleiman, *Irreducibility of the compactified Jacobian*, in *real and complex singularities, Oslo 1976*, Proc. 9th Nordic Summer School/NAVF, P. Holm (ed.), Sijthoff and Noordhoff (1977), 1–12. MR0498546 (58 #16650)
2. A. Altman and S. Kleiman, *Compactifying the Jacobian*, Bull. Amer. Math. Soc. **82** (1976), 947–949. MR0429908 (55 #2917)
3. A. Altman and S. Kleiman, *Compactifying the Picard scheme II*, Amer. J. Math. **101** (1979), 10–41. MR0527824 (81f:14025b)
4. A. Altman and S. Kleiman, *Compactifying the Picard scheme*, Adv. Math. **35** (1980), 50–112. MR0555258 (81f:14025a)
5. A. Beauville, *Counting curves on $K3$ surfaces*, preprint alg-geom/9701019. cf. MR 2000c:14073
6. S. Diaz and J. Harris, *Ideals associated to deformations of singular plane curves*, Trans. Amer. Math. Soc. **309** (1988), 433–468. MR0961600 (89m:14003)
7. C. D'Souza, *Compactification of generalised Jacobians*, Proc. Indian Acad. Sci. **88** (1979), 419–457. MR0569548 (81h:14004)
8. D. Eisenbud, *Homological algebra on complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260**, 35–64 (1980). MR0570778 (82d:13013)
9. E. G. Evans and P. Griffiths, *Syzygies*, London Math. Soc. Lecture Notes Series, **106**, Cambridge Univ. Press (1985). MR0811636 (87b:13001)
10. S. Kleiman and H. Kleppe, *Reducibility of the compactified Jacobian*, Compositio Math. **43** (1981) 277–280. MR0622452 (83f:14027)
11. H. Matsumura, *Commutative Algebra*, Benjamin, N.Y. (1970). MR0266911 (42 #1813)
12. S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math. **77** (1984), 101–116. MR0751133 (85j:14016)
13. C. J. Rego, *The compactified Jacobian*, Ann. Scient. École Norm. Sup. **13**, 211–223 (1980). MR0584085 (81k:14006)
14. M. Schlessinger, *Functors on Artin rings*, Trans. Amer. Math. Soc. **130**, 208–222 (1968). MR0217093 (36 #184)
15. J. R. Strooker, *Homological Questions in Local Algebra*, London Math. Soc. Lecture Notes Series, **145**, Cambridge Univ. Press (1990). MR1074178 (91m:13013)
16. B. Teissier, *Résolution simultanée - I. Famille de courbes*, in *Séminaire sur les Singularités des Surfaces*, M. Demazure, H. Pinkham, B. Teissier, eds., Springer Lecture Notes in Math. **777** (1980). MR0579026 (82d:14021)
17. Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Math. Soc. Lecture Notes Series, **146**, Cambridge Univ. Press (1990). MR1079937 (92b:13016)
18. S. T. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on $K3$*, Nuclear Physics **B471** (1996), 503–512. MR1398633 (97e:14066)